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PENETRATION OF A THIN CYCLICALLY SYMMETRIC THREE-DIMENSIONAL BODY INTO AN ELASTIC HALF-SPACE†

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A method is presented for finding the original of the solution of the problem of the penetration of a thin body with a star-shaped transverse cross-section, obtained in [1] in linear dynamical elasticity theory. The fundamental properties of the solution are discussed. Examples are given of calculations of penetration into a compressible fluid.

1. FINDING THE ORIGINAL OF THE SOLUTION

USING THE principle of angular superposition of the solutions of simpler linear problems a solution was constructed in [1] for the problem of the penetration of a thin body consisting of n symmetric cycles into an elastic half-space. In the case of the penetration of a star-shaped conical body with plane faces and an even number of cycles n, the solution is constructed using the solution of the problem of the penetration with constant velocity v_0 of a thin wing with rhombic profile (Fig. 1). The

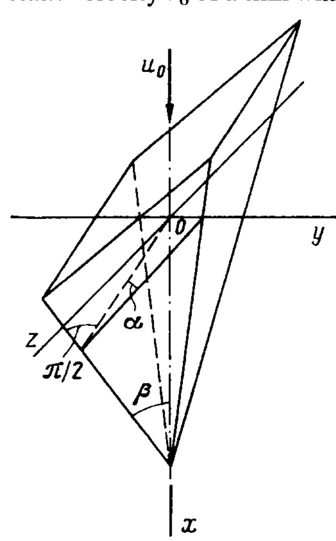

FIG. 1.

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exact solution of this problem was constructed in [1] by transforms of the required functions. In particular, the Laplace transform σ_{yy} of the stress tensor component can be written in the form

$$\frac{\sigma_{yy}}{\rho a^2} = -\frac{k}{p\pi^2}\int_0^{\pi/2}\left\{\int_0^{i\infty}\Phi(r,\varphi)\left[\operatorname{ch}(\psi_1)+\operatorname{ch}(\psi_2)\right]r\,dr\right\}d\varphi \tag{1.1}$$

$$\psi_{1,2} = prR_1\sin(\varphi\pm\theta),\quad R_1=\sqrt{x^2+y^2},\quad \operatorname{tg}\theta = y/z$$

$$\Phi(r,\varphi) = cB_1(r)\,\Delta_3^{-1}\Delta_4^{-1} - d\,[(s^2-m)\,f(M^{-1}) +$$
$$+ q^2 f_1 R^{-1} f(\Delta_1)]\,\Delta_3^{-1}\Delta_4^{-1} - e\,[(s^2-m_1)\,f(M^{-1}) +$$
$$+ q^2 M^{-1}\Delta_2^{-1} f(\Delta_2)]\,\Delta_4^{-1}\Delta_5^{-1} + 2\,[1-2\gamma^2(1-q^2)]\,f_2R^{-1}f(\Delta_1) -$$
$$-2\gamma^2 q^2 f_3 R^{-1} f(\Delta_2),\quad s = r\sin\varphi,\quad q = r\cos\varphi$$

$$B_1(r) = f(M^{-1}) - f(\Delta_1)\,f_1R^{-1},\quad f(w)=\exp(-pxw),\quad \Delta_1=\sqrt{1-r^2}$$

$$\Delta_2=\sqrt{\gamma^{-2}-r^2},\quad \Delta_3=r^2-m,\quad \Delta_4=1-M_t^2s^2,\quad \Delta_5=r^2-m_1$$

$$f_1 = q_1q_2+4M^{-1}r^2\Delta_2,\quad q_1=\gamma^{-2}-2r^2,\quad q_2=\gamma^{-2}-2m$$

$$f_i = f_{i1}+f_{i2}\Delta_4^{-1}\quad (i=2,3),\quad R = q_1^2+4r^2\Delta_1\Delta_2$$

$$f_{21} = -2M_c^2q_3s_2^{-1},\quad f_{22} = -2\,(M_s^2q_3+\gamma^2q_1)\,s_2^{-1},\quad f_{31} = 2M^{-1}M_c^2\,(q_1\Delta_2^{-1} -$$
$$-4\Delta_1\Delta_2^{-1}),\quad f_{32} = 2M\,[\gamma^2m_2\Delta_2\,(q_2-2s_1M^{-2})\,s_1^{-1}+\sin^2\beta\, q_1\Delta_2^{-1} -$$
$$-4M_1^{-2}\Delta_1\,(M_{1s}^2+s_2-1)\,s_2^{-1}]$$

$$s_{1,2} = 1+M\Delta_{1,2},\quad q_3 = r^2+M^{-1}\Delta_2,\quad M_c = M\cos\beta,\quad \gamma = ba^{-1}$$

$$m=\frac{M^2-1}{M^2},\quad m_1=\frac{M_1^2-1}{M^2},\quad m_2=M_{1s}^2-2,\quad M=\frac{v_0}{a},\quad M_1=\frac{v_0}{b}$$

$$a=\sqrt{\frac{\lambda+2\mu}{\rho}},\quad b=\sqrt{\frac{\mu}{\rho}},\quad k=\frac{2\alpha}{\cos\beta},\quad M_t=M\operatorname{tg}\beta,\quad M_{1s}=\frac{M_s}{\gamma},$$
$$M_s = M\sin\beta$$

$$c=\gamma^2(1-2\gamma^2)\,m_2,\quad d=2\gamma^4m_2,\quad e=4\gamma^2$$

Here a and b are, respectively, the velocities of the longitudinal and transverse waves and λ and μ are the Lamé coefficients of the isotropic medium. Terms in the function $\Phi(r, \varphi)$ are grouped so as to show that when $\Delta_3 = \Delta_5 = 0$ there are no poles in $\Phi(r, \varphi)$ because of the structure of the function B_1 and the square brackets. The first term in $\Phi(r, \varphi)$ with the opposite sign and $\gamma = 0$ corresponds to the solution of the problem of the penetration of a body into a compressible liquid, the remaining terms being zero.

We shall find the original of the solution in the first quarter of the y, z plane ($x > 0$), because the perturbed motion of the medium has two planes of symmetry $y = 0$ and $z = 0$ (Fig. 1). The right-hand side of (1.1) can be reduced in various ways to a form coinciding with a direct Laplace transform of the corresponding functions [2], namely

$$\frac{\sigma_{yy}}{\rho a^2} = \int_0^{\infty}(\ldots)\,e^{-p\tau}\,d\tau \tag{1.2}$$

where $\tau = at$ (t is the time) [1].

Here we have chosen an approach in which the wave structures accompanying different body penetration regimes are found explicitly. Below, without loss of generality [3], we shall assume that the Laplace transform parameter $p>0$.

Using the first term of the function $\Phi(r,\varphi)$ as an example we will consider the method of reducing expression (1.1) to the form (1.2), denoting by I that part of the transform σ_{yy} supplied by this term. We introduce a new variable of integration

$$u = \sin\varphi,\ \Delta_4^{-1}d\varphi = A(r,u)\,du,\ A(r,u) = [\sqrt{1-u^2}\,(1-M_t^2r^2u^2)]^{-1} \tag{1.3}$$

Then

$$I = -\frac{kc}{p\pi^2}\int_0^{i\infty}\left\{B(r)\int_0^1 A(r,u)\,[\operatorname{ch}(\psi_3)+\operatorname{ch}(\psi_4)]\,du\right\}dr \tag{1.4}$$

$$B(r) = B_1(r)\,C(r),\ C(r) = r\Delta_3^{-1},\ \psi_{3,4} = pr\,(y\sqrt{1-u^2}\pm zu)$$

Successively deforming the contours of integration in the complex u and r planes, we reduce expression (1.4) to the form (1.2). We find contours in the u plane on which relation

$$y\sqrt{1-u^2}\pm zu = w>0 \tag{1.5}$$

is satisfied in accordance with the expressions for ψ_3 and ψ_4 (1.4).

From (1.5) we find

$$u = \pm(wz - y\sqrt{R_1^2-w^2})\,R_1^{-2},\ y\leqslant w\leqslant R_1 \tag{1.6}$$

$$u = \pm(wz \pm iy\sqrt{w^2-R_1^2})\,R_1^{-2},\ R_1<w<+\infty \tag{1.7}$$

Curves along which (1.5) is satisfied are shown qualitatively in Fig. 2. The segments OA and OA_1 correspond to (1.6) and the curves AB, AB' and A_1B_1, A_1B_1' correspond to (1.7), with $u_A = z/R_1\leqslant 1$. The arrows show the directions in which the parameter w increases. The points C and C_1 with coordinates ±1 are the beginnings of cuts directed along the real axis and associated with the radical $\sqrt{(1-u^2)}$, the argument of which is computed from the formulae

$$\arg(\sqrt{1-u^2}) = (\varphi_1+\varphi_2)/2 \tag{1.8}$$

The angular coefficients of the asymptotes of hyperbolae $B'AB$ and $B_1'A_1B_1$ are, according to (1.7),

$$k' = \pm y/z \tag{1.9}$$

The points E_1 and E_2 (Fig. 2) with coordinates $u_{1,2} = \mp(M_tr)^{-1}$ are the poles of the function $A(r,u)$ (1.3).

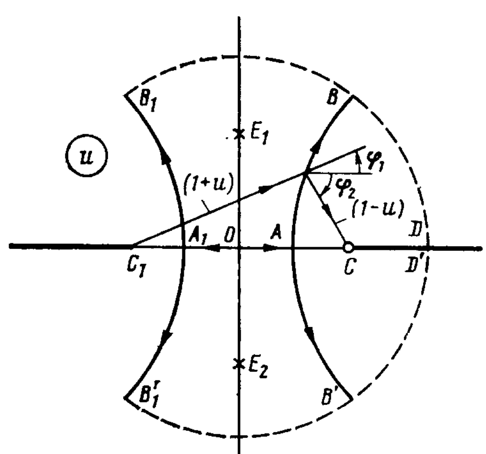

FIG. 2.

The inner integral in (1.4) is taken along the interval OC of the real semi-axis of the complex u plane. Decomposing it into four integrals for each exponential with argument $\pm\psi_3$ or $\pm\psi_4$ inside the square bracket of the integrand, and closing the contour of each of these along $OCDBAO$, $OCD'B'AO$, $OCDBB_1A_1O$, $OCD'B'B_1'A_1O$, respectively, (Fig. 2), where CD and CD' are the upper and lower sides of the cut beginning at the point C, we find using (1.8) that the sum of the integrals along the intervals OA, OA_1 and CD, CD' is zero. The integrals along the arcs DB, DB_1, $D'B'$ and $D'B_1'$ of circles with radii tending to infinity also vanish, because for any $r \in (0, +i\infty)$ and using (1.8) and (1.9), the relations

$$\operatorname{Re}(\pm\psi_3) > 0, \ |\psi_3| \to \infty; \ \operatorname{Re}(\pm\psi_4) > 0, \ |\psi_4| \to \infty$$

are satisfied on them.

As a result of the specified transformations and changing the order of integration we find the following expression for I (1.4):

$$I = -\frac{kc}{2p\pi^2}\left\{\int_{AB} [I_1(u) - I_2(u)]\, du + \int_{AB'} [I_2(u) - I_1(u)]\, du + I_3\right\} \tag{1.10}$$

$$I_1(u) = \int_0^{i\infty} D(r, u)\, dr, \ I_2(u) = \int_0^{-i\infty} D(r, u)\, dr, \ I_3 = 2\pi i \int_0^{i\infty} B(r)\,[\operatorname{res} f(u_1) - \operatorname{res} f(u_2)]\, dr$$

$$\operatorname{res} f(u_{1,2}) = \mp i\,(1 - M_t^2 r^2)^{-1/2} \exp\,[-p\,(\pm iy\sqrt{1 - M_t^2 r^2} + z)/M_t]/2$$

$$D(r, u) = A(r, u)\, B(r) \exp(-prw)$$

In (1.10) w is specified by (1.5) with the plus sign, and $\operatorname{res} f(u_1)$ and $\operatorname{res} f(u_2)$ are residues of the integrand at the poles u_1 and u_2 (the points E_1 and E_2 on Fig. 2).

We will find the contours in the complex r plane in which the relation

$$x\Delta_1 + rw = \tau > 0 \tag{1.11}$$

is satisfied.

It follows from (1.11) that

$$r = (\tau w - x\sqrt{\tau_0^2 - \tau^2})\,\tau_0^{-2}, \ x \leqslant \tau \leqslant \tau_0, \ \tau_0 = \sqrt{w^2 + x^2} \tag{1.12}$$

$$r = (\tau w \pm ix\sqrt{\tau^2 - \tau_0^2})\,\tau_0^{-2}, \ \tau_0 < \tau < +\infty \tag{1.13}$$

Figure 3 qualitatively depicts the curves on which relation (1.11) is satisfied. The segment OE corresponds to

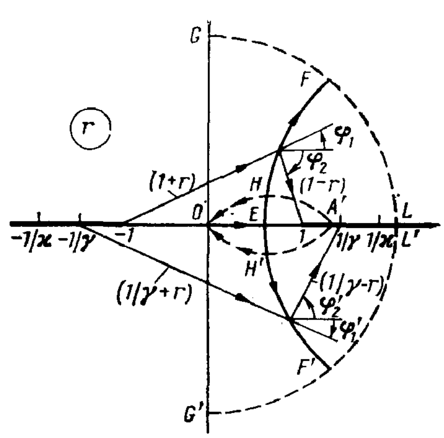

FIG. 3.

(1.12), and the hyperbola branch $F'EF$ corresponds to (1.13). The arrows show the direction of increasing parameter τ. The point E has the coordinate

$$r_e = w\tau_0^{-1} \leqslant 1 \tag{1.14}$$

Also shown are cuts directed along the real axis starting at points with coordinates ± 1 and $\pm\gamma^{-1}$ corresponding to the radicals Δ_1 and Δ_2 occurring in f_1 and R (1.1) and the poles $\pm\varkappa^{-1}$ of the integrand $D(r, u)$ corresponding to a zero of the Rayleigh equation $R = 0$. According to Fig. 3 the arguments of the radical Δ_1 and Δ_2 are calculated using the formulae

$$\arg \Delta_1 = (\varphi_1 + \varphi_2)/2, \ \arg \Delta_2 = (\varphi_1' + \varphi_2')/2 \tag{1.15}$$

We will change from integrating along the imaginary semi-axes OG (OG') in the inner integrals of (1.10) to integrating along the real semi-axes OL (OL') with integrands containing $\exp[-p(x/M + rw)]$, and along the contours OEF (OEF') for integrands containing $\exp[-p(x\Delta_1 + rw)]$. This can be done because in the first case on the circular arc GL ($G'L'$) $\mathrm{Re}(x/M + rw) > 0$, and in the second because on the circular arcs GF ($G'F'$) $\mathrm{Re}(x\Delta_1 + rw) > 0$, and when the radius of the latter tends to infinity the integrals along it of the corresponding functions are equal to zero. For the specified deformations of the contours of integration one must take into account the existence in the right half-plane of poles of the function $C((r)$ (not shown in Fig. 3)

$$r_0 = \sqrt{m}, \ M > 1 \tag{1.16}$$

and lines of poles of the function $A(r, u)$

$$r_{1,2} = (M_t u)^{-1} \tag{1.17}$$

The latter are shown in Fig. 3 by the broken lines. The point A' is the image of the point A (Fig. 2) and has the coordinate

$$r_a = (M_t \cos\theta)^{-1} \tag{1.18}$$

In (1.17) we have used the notation r_1 for the curve $A'HO$, which is the image of the curve AB' (Fig. 2), and r_2 for the curve $A'H'O$ which is the image of the curve AB. The arrows on $A'HO$ and $A'H'O$ show the direction of increasing parameter w.

Thus, when changing the contour of integration OG (OG') on OL (OL'), if $u \in AB'(AB)$ [see (1.10)], and also OG (OG') on OEF (OEF'), one must take account of residues of the corresponding integrand at the poles (1.17), in the first case for all $w \in [R_1, +\infty)$, and in the second for $w \in [w_1, +\infty)$. The parameter values $w = w_1$ and $\tau = \tau_1$ correspond to the point H (H') of intersection of the curves EF (EF') (1.13), $A'HO$ ($A'H'O$) (1.7), (1.17) and are given by the expressions

$$\begin{gathered} w_1^2 = y^2 + R_1^2(z^2 - y^2 + \sqrt{R_1^4 + 4M_t^2x^2z^2})(y^2 + M_t^2x^2)^{-1}/2 \\ \tau_1 = z(w_1^2 + x^2)(w_1^2 - y^2)^{-1}/M_t \end{gathered} \tag{1.19}$$

The point H (H') will exist if $r_e < r_a$, which, in accordance with (1.7), (1.14) and (1.18), leads to the condition

$$\rho_2 \equiv \rho_1/z > M_t, \ \rho_1 = \sqrt{x^2 + y^2 + z^2} \tag{1.20}$$

We note that the integrals along OL (OL') and OE corresponding to (1.10) and (1.15) are of real functions and take opposite signs. Hence they mutually cancel [see (1.10)], except for the first, if the pole r_0 (1.16) is situated to the right of the point E (Fig. 3), which according to (1.14) and (1.16) corresponds to the inequality

$$w < xm_3, \ m_3 = \sqrt{M^2 - 1} \tag{1.21}$$

When inequality (1.21) is satisfied, the difference between the integrals along OL and OL' gives the residue of the integrand at the pole r_0. (In this case the contours OL and OL' should include semicircular arcs centred on r_0 with radius tending to zero.)

After performing the specified transformations and using conditions (1.16), (1.20) and (1.21), we find the following expressions for I (1.10):

$$I = -\frac{kc}{2p\pi^2}\left\{\int_{AB}[I_4(w) - I_5(w)]\,dw + \int_{AB'}[I_5(w) - I_4(w)]\,dw + \right. \tag{1.22}$$

$$\left. + \pi i\left[\int_{AB'}[E_1(u) - E_2(u) + E_3(u)]\,du - \int_{AB}[E_1(u) - E_2(u) + E_3(u)]\,du\right] + I_3\right\}$$

$$I_4(w) = \int_{EF'} E(w,\tau)\,e^{-p\tau}\,d\tau, \quad I_5(w) = \int_{EF} E(w,\tau)\,e^{-p\tau}\,d\tau$$

$$E(w,\tau) = H(w - R_1)\,H(\tau - \tau_0)\,C(r)\,A(r, u)\,f_1 R^{-1}(du/dw)(dr/d\tau)$$

$$E_1(u) = H(M-1)\,H(w - R_1)\,H(xm_3 - w)\,A(r_0, u)\exp[-p(x + wm_3)/M]$$

$$E_2(u) = H(w - R_1)\,A(r_0, u)\exp[-p(xM^{-1} + (y\sqrt{1-u^2}/u + z)\,M_t^{-1})]$$

$$E_3(u) = [H(\rho_3)\,H(w - R_1) + H(-\rho_3)\,H(w - w_1)]\,A(r_0, u)\,f_1(r_1)\times$$

$$\times R^{-1}(r_1)\exp[-p(xM^{-1}\sqrt{1-r_1^2} + (y\sqrt{1-u^2}/u + z)\,M_t^{-1}],\ \rho_3 = M_t - \rho_2$$

Without presenting the calculations, we note that by changing to a new variable of integration v given by

$$\sqrt{1-u^2}/u = v, \quad i\sqrt{1 - M_t^2 r^2} = v \tag{1.23}$$

the single integrals containing the functions $E_2(u)$, $E_3(u)$ and $B(r)$ can be reduced to the difference of two pairs of integrals of the following two functions:

$$E_4(v) = E_6(v)\exp[-p(xM^{-1} + (yv + z)\,M_t^{-1}],\ E_6(v) = (1 - mM_t^2 + v^2)^{-1}$$
$$E_5(v) = E_6(v)\,f_1(v)\,R^{-1}(v)\exp[-p(x\sqrt{M_t^2 - 1 - v^2} + yv + z)/M_t] \tag{1.24}$$

The first pair of integrals of the function $E_4(v)$ are taken along the contours MNK and $MN'K'$ (Fig. 4), where the curve MN (MN') is, according to (1.23), the image of the curve AB' (AB) in the complex v plane. Here $v_N = i$ is the coordinate of the point N, while the coordinate of the point M and the equations of the curves MN and MN' are given by the relations

$$v_M = y/z;\ v = a_1 \pm ib_1$$
$$a_1 = yz/(w^2 - y^2),\ b_1 = w\sqrt{w^2 - R_1^2}/(w^2 - y^2) \tag{1.25}$$

The arrows in Fig. 4 show the direction of increasing parameter $w \in [R_1, +\infty)$. The difference between these

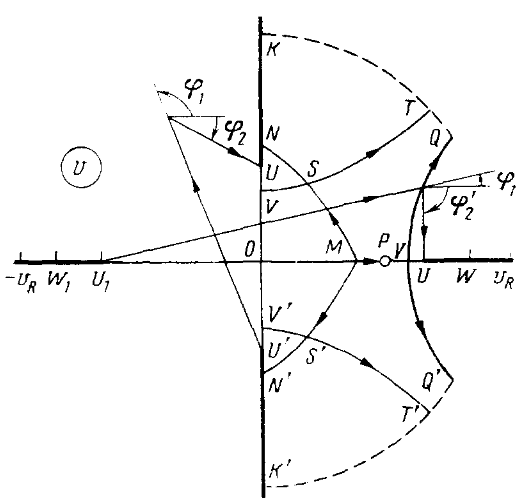

FIG. 4.

integrals gives the residue of the function $E_4(v)$ at the pole P of the function $E_6(v)$ (1.24) when it is to the right of the point M:

$$v_P > v_M,\ v_P = m_s/\cos\beta,\ m_s = \sqrt{M_s^2 - 1},\ M_s > 1 \tag{1.26}$$

The second pair of integrals of the function $E_5(v)$ is taken, in accordance with the arguments of the Heaviside function in $E_3(u)$ (1.22), either along the contours SNK and $S'N'K'$ for $\rho_2 > M_t$, where the point S (S') is the image of the point H (H') (Fig. 3) or along the contours MNK and $MN'K'$ when $\rho_2 < M_t$.

The function $E_5(v)$ contains the radicals $r_3 \equiv \sqrt{(M_t^2 - 1 - v^2)}$ and $r_4 \equiv \sqrt{(M_{1t}^2 - 1 - v^2)}$ (where $M_{1t} = \gamma^{-1}M_t$). If, for example, $M_t < 1$, then the cuts corresponding to the first radical are situated on the imaginary axis beginning at the points U and U' in the interval NN'. The rule for computing the argument of the radical is extended into the left half-plane (Fig. 4): $\arg(r_3) = (\varphi_1 + \varphi_2)/2$. From this it follows that for consistency with relations (1.15) and (1.23) this radical should have a minus sign if v belongs to the first quarter. The same also applies to the case $M_{1t} < 1$. For $M_t > 1$ the cuts corresponding to both radicals are situated along the real axis beginning at the points U (U_1) and W (W_1) in Fig. 4 with coordinates $\pm m_t$ and $\pm m_{1t}$, respectively, [where $m_t = \sqrt{(M_t^2 - 1)}$ and $m_{1t} = \sqrt{(M_{1t}^2 - 1)}$]. According to the rule introduced for computing the argument of the radical in Fig. 4 $\arg(r_3) = (\varphi_1' + \varphi_2')/2$, which leads to a result consistent with relations (1.15) and (1.23), and means that under the specified conditions both radicals are taken with the plus sign. Figure 4 also shows the positions of the pole v_R of the function $F_5(V)$ corresponding to the zero of the Rayleigh equation $R(v) = 0$,

$$v_R = \sqrt{M_R^2\,\mathrm{tg}^2\beta - 1},\quad M_R = M/\varkappa$$

To obtain the difference of the integrals of the function $E_5(v)$ in the form (1.2) we find curves in the v plane along which the relation

$$xr_3 + yv + z = M_t\tau > 0 \tag{1.27}$$

is satisfied.

From (1.27) we find

$$v = (ys_3 - x\sqrt{m_t^2R_1^2 - s_3^2})\,R_1^{-2},\quad s_3 = M_t\tau - z$$

$$M_t > t,\ (xm_t + z)/M_t \leqslant \tau \leqslant \tau_2,\ \tau_2 = (z + m_tR_1)/M_t \tag{1.28}$$

$$v = (ys_3 \pm ix\sqrt{s_3^2 - (M_t^2 - 1)\,R_1^2})\,R_1^{-2}$$

$$M_t > 1,\ \tau_2 < \tau < +\infty;\ M_t < 1,\ z/M_t < \tau < +\infty \tag{1.29}$$

For $M_t < 1$, relation (1.29) corresponds to hyperbola branches VT and $V'T'$ (Fig. 4), passing through the points S and S', where according to (1.25) and (1.29) the parameters w and t take values identical with (1.19). This also occurs for $M_t > 1$ if the point $Y(\tau = \tau_2)$ with coordinates

$$v_Y = ym_t/R_1 \leqslant m_t \tag{1.30}$$

is to the left of the point M, which, according to (1.25) and (1.30), corresponds to condition (1.20). In this case one can pass from integrating the function $E_5(v)$ over to the contours SNK and $S'N'K'$ to integrating over the contours ST and $S'T'$ ($\tau_1 \leqslant \tau < +\infty$), because the integrals over the circular arcs KT and $K'T'$ with radii tending to infinity vanish.

When condition (1.20) is violated the point Y is to the right of the point M (Fig. 4). Here one can pass from integrating over the contours MNK and $MN'K'$ to integrating over the contours MYQ and MYQ' and going round the pole P of the function $E_6(v)$ (1.24) if condition (1.26) is satisfied:

$$v_P < v_Y \tag{1.31}$$

Consequently, when inequalities (1.26) and (1.31) are satisfied, the difference of the integrals of the function $E_5(v)$ over the contours MNK and $MN'K'$ is equal to the sum of the residue of the function $E_5(v)$ at the pole P and the difference of the integrals along the contours YQ and YQ' (1.29).

By introducing the notation

$$(x + wm_3)/M = \tau, \ \tau > (x + R_1 m_3)/M \equiv \tau_3 \tag{1.32}$$

in accordance with the form of the argument of the exponent in the function $E_1(u)$ (1.22) we complete the necessary transformations of (1.22), enabling us to find the original I directly. It remains to say that in all pairs of integrals the integrands take complex-conjugate values for equal values of the parameter τ. Hence their difference is equal to twice the imaginary part of one of the integrals.

The transformation of the remaining terms of the function $\Phi(r, \varphi)$ (1.1) does not present any significant differences.

To sum up, the original of σ_{yy} has the form

$$\begin{aligned}
\frac{\sigma_{yy}}{\alpha\rho a^2} = &- H(M_{s1}) H(v_P - v_M) H(v_Y - v_P) H(\tau - \tau_4) \frac{\gamma^2 m_2^2}{M_{1s}^2 m_s} - \\
&- H(M_{s1}') H(v_P' - v_M) H(v_Y' - v_P') H(\tau - \tau_4') \frac{m_s'}{M_{1s}^2} - H(M-1) H(\tau - \\
&- \tau_3) k_0 \int_{\tau_3}^{\tau} H(Mx - \tau') [cF_1(c_1) + dr_0^2 F_2(c_1)] \left(\frac{dw}{d\tau'}\right) d\tau' - H(M_1 - 1) \times \\
&\times H(\tau - \tau_3') k_1 \int_{\tau_3'}^{\tau} H(M_1 x - \tau') F_2(c_2) \left(\frac{dw'}{d\tau}\right) d\tau' - \\
&- k_0 \operatorname{Im} [H(\rho_3) H(\tau - \tau_2) I_6(\tau_2) + H(-\rho_3) H(\tau - \tau_1) I_6(\tau_1)] - \\
&- k_2 \operatorname{Im} [H(\rho_3') H(\tau - \tau_2') I_7(\tau_2') + H(-\rho_3') H(\tau - \tau_1') I_7(\tau_1')] - \\
&- \frac{2k_0}{\pi} \operatorname{Im} \Big[H(\tau - \rho_1) \int_{R_1}^{\sqrt{\tau^2 - x^2}} \Big[\int_{\tau_0}^{\tau} F_5(r, w) \left(\frac{dr}{d\tau}\right) d\tau' \Big] dw + \\
&+ 2\gamma^2 H(\gamma\tau - \rho_1) \int_{R_1}^{\sqrt{\gamma^2\tau^2 - x^2}} \Big[\int_{\tau_0'}^{\tau} F_6(r', w) \left(\frac{dr'}{d\tau}\right) d\tau' \Big] dw \Big] - \\
&- \frac{4\gamma^2 k_0}{\pi} \operatorname{Im} [H(R_1 - x_1) H(\tau - R_1 - x_3) I_8(R_1) + \\
&+ H(x_1 - R_1) H(\tau - x_2) I_8(x_1)] - H(M_t - 1) H(\rho_3') H(M_t y - m_t x_1) \times \\
&\times \operatorname{Im} [H(m_t - v_M) H(\tau - \tau_5) I_9(\tau_5) + H(v_M - m_t) H(\tau - \tau_6) I_9(\tau_6)]
\end{aligned} \tag{1.33}$$

$$I_6(u) = \int_u^{\tau} F_3(v) \left(\frac{dv}{d\tau'}\right) d\tau', \quad I_7(u) = \int_u^{\tau} F_4(v') \left(\frac{dv'}{d\tau}\right) d\tau'$$

$$I_8(u) = \int_u^{\tau - x_3} \Big[\int_{u + x_3}^{\tau} H(\tau_0' - \tau') F_7(r', w) \left(\frac{dr'}{d\tau'}\right) d\tau' \Big] dw,$$

$$I_9(u) = \int_u^{\tau} H(\tau_2' - \tau') F_8(v') \left(\frac{dv'}{d\tau'}\right) d\tau'$$

$$\tau_4 = xM^{-1} + (yv_P + z) M_t^{-1}, \quad M_{s1} = M_s - 1, \quad k_0 = \frac{k}{\pi\alpha},$$

$$k_1 = 4\gamma^4 k_0 m_1, \quad k_2 = 2\gamma^4 k_0 M_t^{-4}$$

$$\tau_5 = (ym_t + z) M_t^{-1} + x_3, \quad \tau_6 = (x \sqrt{M_{1t}^2 - R_1^2/z^2} + R_1^2/z) M_t^{-1}$$

$$F_1(u) = A_1(w) A_2(u) (ue_1 + e_2^2), \quad F_2(u) = A_1(w) A_2(u) [u(e_1 -$$

$$- 1) + e_2^2]$$

$$A_1(w) = -(da_1/dw)\,e_2^{-1}, \quad A_2(u) = (u^2 + e_2^2)^{-1}, \quad e_1 = 1 + a_1^2 - b_1^2,$$

$$e_2 = 2a_1 b_1$$

$$F_3(v) = \{(c + dv^2 M_t^{-2})\,f_1(v)\,E_6(v) - 2\,[1 - 2\gamma^2(1 - v^2 M_t^{-2})] \times f_{22}\cdot$$

$$\cdot (v)\,M_t^{-2}\}\,R^{-1}(v)$$

$$F_4(v) = F_8(v) + 2\operatorname{tg}\beta M_{1t}^2 v^2 (1 - m_1 M_t^2 + v^2)^{-1} r_4^{-1}$$

$$F_5(r, w) = \{[cF_1(d_1) + dr^2 F_2(d_1)]\,C(r)\,f_1 - 2r\,[[(1 - 2\gamma^2)\,F_1(d_1) +$$

$$+ 2\gamma^2 r^2 F_2(d_1)]\,f_{22} + [(1 - 2\gamma^2)\,A_1(w) + 2\gamma^2 r^2 F_2(e_1)]\,f_{21}]\}\,R^{-1},$$

$$F_6(r, w) = 2r^3 M_1^{-1}(r^2 - \gamma^2 m_1)^{-1} \Delta_1^{-1} F_2(d_2) + F_7(r, w)$$

$$F_7(r, w) = r^3\,[f_{31}' F_2(e_1) + f_{32}' F_2(d_2)]\,\gamma^{-4}/R', \qquad F_8(v) = v^2\gamma^{-4} f_{32}'(v)/R'(v)$$

$$c_1 = e_1 - mM_t^2, \quad d_1 = e_1 - r^2 M_t^2, \quad x_1 = \gamma x / \sqrt{1 - \gamma^2}, \quad x_2 = \gamma^{-2} x_1,$$

$$x_3 = x_2 (1 - \gamma^2)^{-1}$$

In (1.33) the quantities M_{s1}', ρ_3', c_2 and d_2 have expressions that are the same as M_{s1}, ρ_3, c_1 and d_1 except that M is replaced by M_1 while m_s', v', v_Y', τ_1', τ_2', τ_3', τ_4' and w' are the same as the corresponding unprimed quantities, but with the specified change made in the sign of the radical. The variable r' is given by expressions (1.12) for $\tau' \leq \tau_0' \equiv \gamma^{-1}\tau_0$ and (1.13) with the plus sign for $\tau' > \tau_0'$, in both of which τ should be replaced by $\gamma\tau'$. The variable v' is given by expressions (1.28) for $\tau' \leq \tau_2'$ and (1.29) with the plus sign for $\tau > \tau_1'$ and $\tau' > \tau_2'$, in both of which τ in s_3 should be replaced by τ' and under the root sign M should be replaced by M_1. The functions f_{31}', f_{32}' and R' are similar to f_{31}, f_{32} and R, but their argument is $\gamma^{-1}r$. In (1.33) r and v are given by expressions (1.13) and (1.29) with the plus sign, while w in the integral with factor $H(M-1)$ is given by (1.32). The functions occurring in (1.1) and written in (1.33) and above with argument v or v' are obtained from the originals after the substitution (1.23) or $i\sqrt{(1 - M_{1t}^2 r^2)} = v'$.

2. ANALYSIS OF THE SOLUTION

First we note that because of the conical shape of the penetrating body the solution (1.33) depends only on the self-similar variables x/τ, y/τ and z/τ.

The arguments of the Heaviside function determine the conditions (flow regime) and domain of existence of each part of the solution. The first pair of terms corresponds to the supersonic nature of the motion of the sharp leading edges of the body with respect to the velocity of longitudinal and transverse waves and introduces constant components into σ_{yy}.

According to the expressions for τ_4 and τ_4' (1.33) the domain of definition of these components is bounded by plane waves attached to the leading edge, and according to the arguments of other Heaviside functions, by planes perpendicular to the specified plane fronts and passing through the x and z axes. The plane waves are touched by conical waves $\tau = \tau_3$, $\tau = \tau_2$ and $\tau = \tau_3'$, $\tau = \tau_2'$ determining, together with other arguments of the Heaviside functions, the domain of influence of terms corresponding to supersonic motion of the point of the body ($M > 1$, $M_t > 1$) and its edges through the free surface ($M_t > 1$, $M_{t1} > 1$) and its edges for both supersonic and subsonic motion of the leading edges.

The terms in (1.33) containing τ_1 and τ_1' in the Heaviside function arguments can be related to the influence of the motion of a leading edge through the free surface for subsonic velocities of its displacement along the surface $x = 0$ and partially for supersonic velocities, according to the arguments of the associated Heaviside functions.

The seventh term in (1.33) describes perturbed motion inside the spherical fronts of longitudinal and transverse waves. The remaining terms give the influence of transverse waves associated with the reflection of longitudinal waves from the free surface. The eighth term is due to reflection of a spherical longitudinal wave from the free surface. The boundaries of its domain of existence will be a conical transverse wave $\tau = R_1 + x_3$ touching the spherical front of the transverse wave and the plane $\tau = x_2$ crossing the specified line of contact.

The last term in (1.33) occurs only in the case when $M_t > 1$ and is due to the reflection of the conical longitudinal wave $\tau = \tau_2$ corresponding to this type of motion from the free surface. The boundaries of the domain of definition of this part of the solution will be the plane transverse wave $\tau = \tau_5$ touching the conical transverse wave $\tau = \tau_2'$, the surface $\tau = \tau_6$, continuously extended into the interior of the domain, and other surfaces specified by the arguments of the associated Heaviside functions.

We will consider characteristic properties of the stress tensor component σ_{yy} in the perturbed region. According to the structure of the integrands of the third, fourth, fifth and sixth terms in (1.33), σ_{yy} has a logarithmic singularity at the leading edge of the body. One can relatively easily draw this conclusion for the motion regime $M > 1$, $M_1 > 1$ so long as the third and fourth terms are integrable. For other motion regimes one must consider the asymptotic behaviour of the corresponding integrals. In all cases we have the following limiting relation in the plane $y = 0$:

$$M\tau - x \to z \operatorname{ctg} \beta$$

$$\frac{\sigma_{yy}}{\rho a^2} = \lim_{g \to \infty} \left\{ -\frac{\alpha \gamma^4}{2\pi M_s^2 \sqrt{1 - M_s^2}} \left[m_2^2 - 4 \sqrt{1 - M_s^2} \sqrt{1 - M_{1s}^2} \right] \ln g \right\} \tag{2.1}$$

The expression in square brackets is the left-hand side of Rayleigh's equation with argument M_s^{-1}. If $M_s < \varkappa$ (where $\varkappa$ is as before the dimensionless velocity of the surface waves), then this expression is negative, while in the case when $M_s > \varkappa$ it is positive. Consequently, when the velocity of the leading edges is less than the Rayleigh wave velocity, the normal stress on the surface of the body in a neighbourhood of the edges takes large positive values, which means that the contact of the penetrating body with the medium in the neighbourhood of the leading edges will be broken, and a crack will penetrate in front of them. For $M_s > \varkappa$ there will be a large negative normal stress in front of the edges and the contact between the body and the medium will not be broken.

Solution (1.33) found for the case of a body (Fig. 1) penetrating into an elastic medium has yet another logarithmic singularity. By making a point in the perturbed medium tend to the axis of the body we obtain the limiting relation

$$M\tau > x, \quad R_1 \to 0$$

$$\sigma_{yy}/(\rho a^2) = \lim_{g \to \infty} \{ -(2\alpha\gamma^2 \cos \beta/\pi) \ln g \} \tag{2.2}$$

It follows from (2.2) that in a neighbourhood of an edge of the penetrating body ($z = 0$) there appears a large normal stress. We note that one can arrive at this conclusion directly using expression (1.4), putting $y = z = 0$. Investigations of the stress tensor component σ_{zz} showed that when $R_1 \to 0$, $\sigma_{zz} \to -\infty$ following the same law as σ_{yy} (2.2). The existence of this singularity in the solution is due to the fracture of the elastic medium element at the edge of the body in the plane $z = 0$.

3. PENETRATION INTO A LIQUID

Solution (1.33) for σ_{yy} with $\mu = \gamma = 0$ taken with the opposite sign is a perturbation of the pressure in the problem of the penetration of a body (Fig. 1) with constant velocity into a compressible fluid [3]. In this case the single integrals in (1.33), containing fundamental singularities of the solution, can be integrated. Omitting the complicated calculations, we will write the expression for the pressure referred to a dimensionless area σ of half the centre section S_M of the penetrating body (the section coinciding with the free surface of the liquid $x = 0$): $\sigma = S_M(t)/(2v_0^2t^2) = \alpha \operatorname{tg}\beta \sin\beta$, in self-similar variables $\bar{x} = x/\tau$, $\bar{y} = y/\tau$, $\bar{z} = z/\tau$ (the bars are henceforth omitted)

$$
\begin{aligned}
P &= H(M_{s1})\, H(v_P - v_M)\, H(v_Y - v_P)\, H(1-\tau_4)\, \Pi + H(M-1) \times \\
&\times H(xm_3 - R_1)\, H(1-\tau_3)\, \{H(-M_{s1})\, \Pi_1\, [H(M_x) \ln|\chi^+(1)/\chi^-(1)| + \\
&+ H(-M_x) \ln|\chi^+(Mx)/\chi^-(Mx)|] + H(M_{s1})(\Pi/\pi)[H(M_x) \times \\
&\times [f_4(p_{03}(1)) - f_4(p_{04}(1))] + H(-M_x)[f_4(p_{03}(Mx)) - f_4(p_{04}(Mx))]\} + \\
&+ [H(\rho_3)\, H(1-\tau_2) + H(-\rho_3)\, H(1-\tau_1)] \times \\
&\times \{H(-M_{s1})\, \Pi_1 \ln|\chi_1^+(1)/\chi_1^-(1)| - H(M_{s1})(\Pi/\pi)[f_4(p_{58}^{-1}(1)) + \\
&+ f_4(p_{59}^{-1}(1))]\} - H(-\rho_3)\, H(1-\tau_1)\{H(-M_{s1})\, \Pi_1 \ln|\chi_1^+(\tau_1)/\chi_1^-(\tau_1)| - \\
&- H(M_{s1})(\Pi/\pi)[f_4(p_{58}^{-1}(\tau_1)) + f_4(p_{59}^{-1}(\tau_1))]\} + \\
&+ H(1-\rho_1) \frac{2k_0 \operatorname{ctg}\beta}{\pi \sin\beta} \operatorname{Im} \int\limits_{R_1}^{\sqrt{1-x^2}} \left[\int\limits_{\tau_0}^{1} F_5(r, w) \left(\frac{dr}{d\tau'}\right) d\tau'\right] dw
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
&\Pi = M^2 v_p^{-1}, \quad \Pi_1 = M^2/(2\pi\Pi_2), \quad \Pi_2 = \sqrt{1 - M_s^2}/\cos\beta \\
&\chi^{\pm}(\tau) = [\chi_0(\tau) \pm \chi_1]^2 + \chi_2, \quad \chi_0(\tau) = [(M\tau - x)^2 - m_3^2 R_1^2]^{1/2}/(M\tau - x) \\
&\chi_1 = \Pi_2 R_1^2/(z^2 + y^2\Pi_2^2), \quad \chi_2 = m_3^2 y^2 z^2 \operatorname{tg}^4\beta/(z^2 + y^2\Pi_2^2)^2 \\
&p_{ij}^k(\tau) = \chi_i(\tau)\, \chi_j^k, \quad M_x = Mx - 1, \quad f_4(u) = \operatorname{arctg}(u)| \\
&\chi_1^{\pm}(\tau) = [\chi_5(\tau) \pm \chi_6]^2 + \chi_7, \quad \chi_{3,4} = (z \pm yv_P)/(y \mp zv_P), \quad \chi_6 = x\Pi_2 \\
&\chi_5(\tau) = [s_3^2 + (1 - M_t^2) R_1^2]^{1/2}, \quad \chi_7 = y^2 \operatorname{tg}^2\beta, \quad \chi_{8,9} = xv_P \mp y \operatorname{tg}\beta
\end{aligned}
$$

In (3.1) the simplifications for $F_5(r, w)$ when $\gamma = 0$ are omitted. According to the expressions for χ_3 and χ_8 an indeterminacy appears in the calculation of the pressure in the planes $y - zv_p = 0$ and $xv_p - y \operatorname{tg}\beta = 0$ in the motion regime $M_s > 1$. The corresponding terms in (3.1) should be computed by making points of the field tend to these planes from both directions. This treatment together with the first term in (3.1) ensures the continuity of P. A similar situation occurs in (1.33) with $M_s > 1$ and $M_{1s} > 1$.

We also note that when $\beta \to \pi/2$ solutions (1.33) and (3.1) turn into solutions corresponding to the penetration of a wedge into an elastic half-space and a compressible liquid [4]. Solution (3.1) also contains within itself the solution of [5].

Figures 5 and 6 show the results of calculations of the pressure P for penetration into a compressible fluid of a star-shaped body with four symmetric cycles ($n = 4$) with $M = 0.8$; $\beta = \pi/4$ (Fig. 5) and $\beta = \pi/3$ (Fig. 6). The solution is constructed using the principle of superposition [1] and (3.1) as the basic solution. The solid curves correspond to the pressure distribution in the $y = 0$ plane, and the broken curves to $\theta = \pi/4$. The pairs of curves 1–5 correspond to the sections $x = 0.9$,

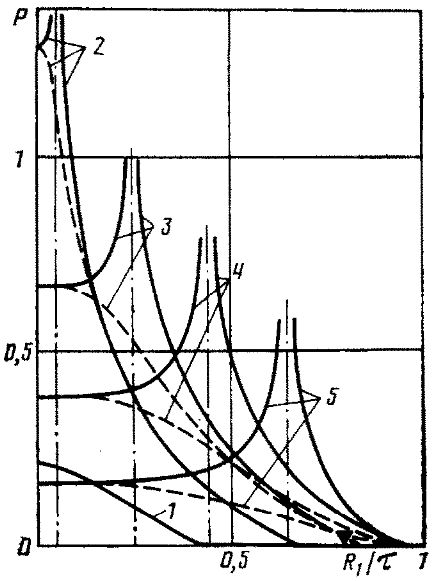

FIG. 5.

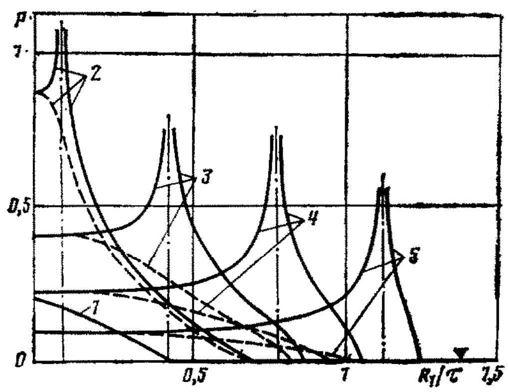

FIG. 6.

0.75, 0.55, 0.35 and 0.15. The vertical dash-dot lines in those sections show the position of the leading edge, in the neighbourhood of which the pressure has a logarithmic singularity. The black triangles show the position of the leading edge on the free surface. In the case $\beta = \pi/3$ (Fig. 6) the leading edge is displaced along the free surface at a supersonic velocity $M_t > 1$. It can be seen that in this penetration regime there is an inflated pressure distribution profile immediately behind the wave front, unlike in the case when $\beta = \pi/4$ ($M_t < 1$) (Fig. 5).

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